Definition: A fraction is a numerical representation for part of a whole.

The DENOMINATOR tells how many equal pieces the whole is divided into.

\[
\begin{array}{c}
\text{1/5} \\
\end{array}
\]

The NUMERATOR tells how many pieces of the whole the fraction represents.

\[
\begin{array}{c}
\text{1} \\
\text{5}
\end{array}
\]

Add all the pieces to get the whole:

\[
\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{5}{5} = 1
\]

**Fact:** \( \frac{x}{x} = 1 \) (\( x \neq 0 \))

The fraction bar represents division \((\div)\), so \( \frac{1}{5} = 0.2\), \( \frac{10}{2} = 5\), and \( \frac{38}{5} = 7.6\).

Any fraction with a Denominator of 1 is equal to its Numerator: \( \frac{x}{1} = x \div 1 = x \)

Division by zero is Undefined, so the Denominator of a fraction can never be zero: \( \frac{x}{y} \) (\( y \neq 0 \))

**Fundamental Property of Fractions . . .**

\[
\frac{ax}{bx} = \frac{a}{b}
\]

*We use this fact when we Reduce (or Simplify) fractions to lowest terms.

\[
\frac{4}{10} \rightarrow \frac{2 \cdot 2}{5 \cdot 2} \rightarrow \frac{2}{5} \cdot \frac{1}{1} = \frac{2}{5}
\]

**Equality of Fractions . . .**

\[
\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc
\]

*We use this fact when we Cross Multiply to solve for an unknown numerator or denominator.

\[
\frac{x}{10} = \frac{2}{5} \rightarrow 5x = 20 \rightarrow \frac{5x}{5} = 20 \rightarrow \frac{x}{5} = 4
\]

**Addition and Subtraction of fractions require a common denominator.**

\[
\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}
\]

When the denominators are different, multiply one or both fractions by another fraction that is the equivalent of \( \frac{1}{1} (\frac{x}{x}) \) to create a Common Denominator; then add or subtract.

\[
\frac{a}{b} + \frac{c}{d} = \frac{a(b)}{b(d)} + \frac{c(b)}{d(b)} = \frac{ad + cb}{bd}
\]

You may be able to multiply the smaller Denominator by something to create the larger one:

\[
\frac{1}{2} + \frac{3}{4} \rightarrow \frac{1}{2} (\frac{2}{2}) + \frac{3}{4} \rightarrow \frac{2}{4} + \frac{3}{4} = \frac{5}{4}
\]

If not, then multiply the two Denominators together:

\[
\frac{1}{2} \cdot \frac{5}{7} \rightarrow \frac{1}{2} \cdot \frac{7}{5} (\frac{7}{2}) \rightarrow \frac{7}{14} - \frac{10}{14} = -\frac{3}{14}
\]

**Multiplication and Division of fractions do not require a common denominator.**

\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}
\]

Note that it is easier to reduce before actually multiplying.

\[
\frac{1}{2} \cdot \frac{5}{3} \cdot \frac{2}{5} = \frac{1 \cdot 6 \cdot 2}{2 \cdot 3 \cdot 5} = \frac{1}{3}
\]

To divide fractions, first invert the Divisor (second fraction) to get its Reciprocal; then multiply.

\[
\frac{a}{b} \div \frac{c}{d} \rightarrow \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}
\]

Factoring before multiplying can help with reducing:

\[
\frac{3}{8} \div \frac{1}{8} \rightarrow \frac{3}{8} \cdot \frac{1}{8} = \frac{3 \cdot 2}{8} = \frac{6}{8} \rightarrow \frac{3 \cdot \sqrt{2} \cdot 1}{\sqrt{4} \cdot \sqrt{2} \cdot 1} = \frac{3}{4}
\]

*But, note that 2 is a factor of 8, so ...*

\[
\frac{3 \cdot \sqrt{2} \cdot 1}{\sqrt{4} \cdot \sqrt{2} \cdot 1} = \frac{3}{4}
\]
A Proper Fraction has a numerator that is smaller than its denominator and represents a quantity less than the whole, or $< 1$:

$\frac{1}{5}, \frac{2}{5}, \frac{3}{5},$ and $\frac{4}{5}$ are proper fractions.

An Improper Fraction has a numerator larger than its denominator and represents a quantity greater than the whole, or $> 1$:

$\frac{6}{5}, \frac{10}{5},$ and $\frac{27}{5}$ are improper.

Mixed numbers, such as $7\frac{3}{5}$, $23\frac{6}{7}$, and $8\frac{110}{241}$, are whole numbers and portions less than 1 (fractions) added together.

$7\frac{3}{5}$ means $7$ and $\frac{3}{5}$, or $7 + \frac{3}{5}$

It is often useful in doing calculations to convert mixed numbers to improper fractions. To do so, change the whole number to a fraction with the same denominator as the other fraction and add:

$$7 = \frac{7}{1} \times \frac{5}{5} = \frac{35}{5}; \text{ then } \frac{35}{5} + \frac{3}{5} = \frac{38}{5}$$

A quick way: \( \text{(WHOLE NUMBER} \times \text{DENOMINATOR} + \text{NUMERATOR})/\text{DENOMINATOR} \):

\( \frac{3}{5} = \frac{7 \times 5 + 3}{5} = \frac{38}{5} \)

To go from improper fraction to mixed number, simply divide the Numerator by the Denominator. The Remainder over the Divisor is the fractional portion.

Comparing fractions . . .

Obviously $\frac{5}{8} > \frac{3}{8}$, but what about $\frac{5}{8}$ and $\frac{7}{12}$? Here’s how to tell:

Express each fraction with a Common Denominator:

$$\frac{\frac{5}{8}}{\frac{3}{3}} = \frac{15}{24} \text{ and } \frac{\frac{7}{12}}{\frac{2}{2}} = \frac{14}{24} \text{ to } \frac{15}{24} \text{ to } \frac{14}{24}. \text{ so } \frac{5}{8} > \frac{7}{12}$$

Or, express each as a decimal:

$$\frac{5}{8} = 0.625 \text{ and } \frac{7}{12} = 0.5833 \rightarrow 0.625 > 0.5833 \ldots$$

Also, test for Equality of Fractions \( \left(\frac{a}{b} = \frac{c}{d} \iff ad = bc\right) \):

$$\frac{5}{8} \cdot \frac{7}{12} \rightarrow 5 \cdot 12 \? 8 \cdot 7 \rightarrow 60 > 56, \text{ so } \frac{5}{8} > \frac{7}{12}$$

Eliminating fractions . . .

A fraction multiplied by its Reciprocal equals 1; use this fact to isolate $x$ and solve an equation:

$$\frac{3}{5} x = 2 \rightarrow \left(\frac{5}{3}\right) \frac{3}{5} x = \left(\frac{5}{3}\right) \frac{2}{1} \rightarrow 1x = \frac{10}{3}$$

so $x = \frac{10}{3}$, or $3.33 \ldots$

Multiply through by the Least Common Multiple (LCM) of the denominators replaces fractions with whole numbers, making an equation easier to work with:

$$\frac{2}{3} x^2 + \frac{5}{6} x = 4 \rightarrow \left(\frac{6}{1}\right) \frac{2}{3} x^2 + \left(\frac{6}{1}\right) \frac{5}{6} x = \left(\frac{6}{1}\right) \frac{4}{1} \rightarrow 4x^2 + 5x = 24$$

From Decimals to Fractions to Percents . . .

Decimals can be expressed as fractions with a Denominator that is a Power of 10. The number of digits behind the decimal tells how many zeros belong in the denominator. Remember to reduce fractions when possible:

$$0.5 = \frac{5}{10} = \frac{1}{2}, 0.25 = \frac{25}{100} = \frac{1}{4}, 0.225 = \frac{225}{1000} = \frac{9}{40} \text{ and } 1.0 = \frac{1}{1} = 1 \text{ (no digits behind the decimal, so } 10^0 = 1)$$

To express a fraction as a percent, first divide the Numerator by the Denominator; then multiply the resulting decimal number by 100 (or, simply move the decimal two places to the right):

$$\frac{1}{2} = .50 = 50\%, \frac{1}{4} = .25 = 25\%, \frac{9}{40} = .225 = 22.5\%, \text{ and } \frac{1}{1} = 1.00 = 100\%$$